Trigonometric Substitution

$\sqrt{a^2 - x^2}$ $\sqrt{a^2 + x^2}$ $\sqrt{x^2 - a^2}$

Type $\sqrt{a^2 - x^2}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\hookrightarrow a^2\sin^2\theta + a^2\cos^2\theta = a^2$$

$$\hookrightarrow (a\sin\theta)^2 + (a\cos\theta)^2 = a^2$$

$$\hookrightarrow (a\cos\theta)^2 = a^2 - (a\sin\theta)^2$$

$$y = \sqrt{y^2} = \sqrt{a^2 - x^2}$$

$$x = a\sin\theta$$

Ex:

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

$\sqrt{4 - x^2} \rightsquigarrow a = 2$

Use $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (on this domain $\cos\theta \geq 0$)

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = \int \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta}(2\cos\theta\, d\theta)$$

$$= \int \frac{\sqrt{4\cos^2\theta}}{4\sin^2\theta}\cdot 2\cos\theta\, d\theta = \int \frac{2\cos\theta}{4\sin^2\theta}\, 2\cos\theta\, d\theta$$

$$= \int \frac{\cancel{4}\cos^2\theta}{\cancel{4}\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\, d\theta$$

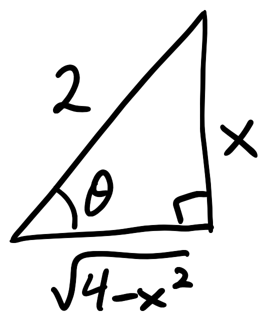
$$= -\cot\theta - \theta + C$$

$$= -\cot\left(\arcsin\frac{x}{2}\right) - \arcsin\frac{x}{2} + C$$

$$= \boxed{\frac{-\sqrt{4-x^2}}{x} - \arcsin\frac{x}{2} + C}$$

$x = 2\sin\theta \rightarrow \theta = \arcsin\frac{x}{2}$

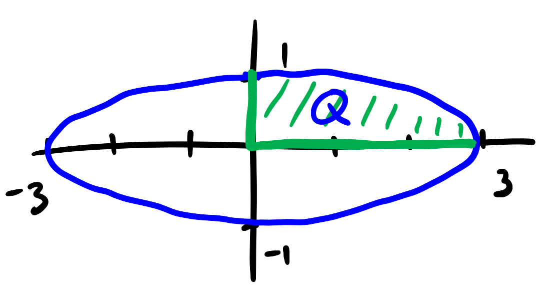
$\frac{x}{2} = \sin\theta$



$\cot\theta = \frac{\sqrt{4-x^2}}{x}$

Ex: Find the area inside the ellipse

$$\frac{x^2}{9} + y^2 = 1.$$



$\times 4$

Top half

$$y = \sqrt{1 - \frac{x^2}{9}}$$

$$Q = \int_0^3 \sqrt{1 - \frac{x^2}{9}}\, dx = \int_0^3 \sqrt{1 - \left(\frac{x}{3}\right)^2}\, dx$$

$$\frac{x}{3} = \sin\theta \rightarrow x = 3\sin\theta$$

$$dx = 3\cos\theta\, d\theta \qquad \boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$$

$$0 = \sin\theta \rightarrow \theta = 0$$

$$\frac{3}{3} = \sin\theta \rightarrow \theta = \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \sqrt{1 - \sin^2\theta}\, (3\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\cos^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 3\cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{3}{2}(1 + \cos 2\theta)\, d\theta$$

$$= \frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{3}{2}\left(\left(\frac{\pi}{2} + \cancel{\frac{1}{2}\sin\pi}\right) - \left(\cancel{0} + \cancel{\frac{1}{2}\sin 0}\right)\right)$$

$$= \frac{3\pi}{4}$$

Area inside the ellipse is $4 \cdot \frac{3\pi}{4} = 3\pi$.

Type $\sqrt{a^2+x^2}$: $a^2 + a^2\tan^2\theta = a^2\sec^2\theta$

$$x = a\tan\theta$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Type $\sqrt{x^2-a^2}$: $a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

$$x = a\sec\theta$$

$$0 \le \theta < \frac{\pi}{2} \quad \text{or} \quad \pi \le \theta < \frac{3\pi}{2}$$

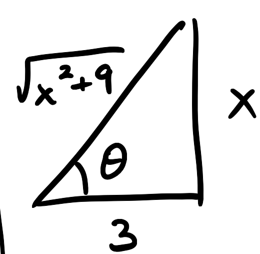
(pos x-values) (neg. x-values)

Ex: $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx$

$x = 3\tan\theta$, $\underbrace{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$

$dx = 3\sec^2\theta\,d\theta$ $\quad$ $\sec\theta > 0$

$\tan\theta = \frac{x}{3}$



$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

$$= \int \frac{27\tan^3\theta}{\sqrt{9+9\tan^2\theta}}\,3\sec^2\theta\,d\theta$$

$$= \int \frac{27\tan^3\theta}{\sqrt{9\sec^2\theta}}\,3\sec^2\theta\,d\theta$$

$$= \int \frac{27\tan^3\theta}{\cancel{3}\,\cancel{\sec\theta}}\,\cancel{3}\sec^{\cancel{2}}\theta\,d\theta = 27\int \tan^3\theta\sec\theta\,d\theta$$

$$= 27\int \tan^2\theta \cdot \underbrace{\tan\theta\sec\theta}\, d\theta$$

$$= 27\int (\sec^2\theta - 1)\,\underbrace{\sec\theta\tan\theta\, d\theta}$$

$u = \sec\theta$

$du = \underbrace{\sec\theta\tan\theta\, d\theta}$

$$= 27\int (u^2 - 1)\, du = 27\left(\frac{1}{3}u^3 - u\right) + C$$

$$= 9\sec^3\theta - 27\sec\theta + C$$

$$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + C$$

$$= \frac{(x^2+9)^{3/2}}{3} - 9\sqrt{x^2+9} + C$$